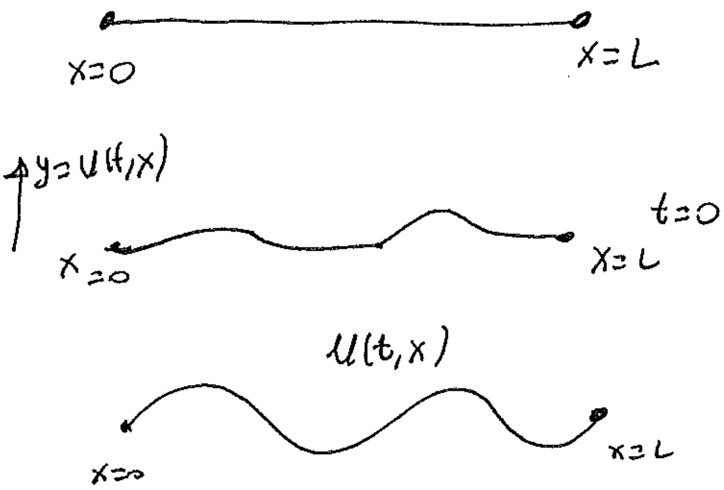
Chapter 4. Wave Equation: Vibrating string and membranes

4.4 Vibrating string with Fixed Ends



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$c^2 = \frac{T_0}{\rho_0}$$

$$\left.\begin{array}{l} T_0 = \text{tension} \\ \rho_0 = \text{density} \end{array}\right\} \text{of the string}$$

$$\text{PDE} \quad : \quad u_{tt} = c^2 u_{xx} \quad , \quad 0 < x < L, \quad t > 0$$

$$\text{BCs} \quad : \quad \left.\begin{array}{l} u(0, t) = 0 \\ u(L, t) = 0 \end{array}\right\} \quad t > 0$$

$$\text{ICs} \quad : \quad \left.\begin{array}{l} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{array}\right\} \quad 0 < x < L$$

Is this a well posed problem.

1) Existence and uniqueness

2) Stability

Solution (the Formal Solution)

Separation of variables

$$u(x,t) = \phi(x)\, h(t)$$

we get

$$\frac{1}{\phi}\frac{d^2\phi}{dx^2} = \frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = -\lambda$$

where λ is the separation of variables

$$\phi'' + \lambda\phi = 0 \quad , \quad h'' + \lambda c^2 h = 0$$

with the homogeneous BCs

$$\phi'' + \lambda\phi = 0 , \quad \phi(0) = 0, \ \phi(L) = 0$$

we remember this eigenvalue problem. solution is

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 , \quad \phi_n = \sin\frac{n\pi}{L}x \quad , \quad n = 1, 2, ..$$

the only possibility is $\lambda > 0$.

Solution of the time dependent part

$$h(t) = A\cos\sqrt{\lambda}ct + B\sin\sqrt{\lambda}ct$$

for each eigen value λ_n we have

$$\phi_n(x,t) = \phi_n(x)\, h_n(t)$$

$$= \sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\frac{n\pi}{L}ct + B_n\sin\frac{n\pi}{L}ct\right]$$

Superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} \Phi_n$$

$$= \sum_{n=1}^{\prime} \sin\left(\frac{n\pi}{L}x\right)\left[A_n \cos\frac{n\pi}{L}ct + B_n \sin\frac{n\pi}{L}ct\right]$$

where A_n and B_n are constants to be determined.

Using the initial conditions.

$$u(x,0) = f(x) = \sum_{n=1}^{\infty\,\prime} A_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x \, dx}$$

$$u_t(x,0) = g(x) = \sum_{n=1} \frac{n\pi c}{L} B_n \sin\frac{n\pi}{L}x$$

$$\boxed{B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\frac{n\pi}{L}x \, dx}$$

a) Existence:

i) u_{xx} :

$$\left| \sum \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L}x\right)\left[A_n \cos\frac{n\pi}{L}ct + B_n \sin\frac{n\pi}{L}ct \right|^2$$

$$\leq \sum \left(\frac{n\pi}{L}\right)^2 \left(|A_n| + |B_n| \right)$$

hence $\sum n^2 |A_n|$ and $\sum n^2 |B_n|$

must be convergent.

for this purpose

$$A_n = \frac{2}{L}\left[-\frac{L}{n\pi} f(x) \cos\frac{n\pi}{L}x \Big|_0^L + \frac{L}{n\pi}\int_0^L f' \cos\frac{n\pi}{L}x \, dx \right.$$

$f(0) = f(L) = 0$

$$A_n = \frac{2}{n\pi}\left[\cancelto{0}{\frac{L}{n\pi} f' \sin\frac{n\pi}{L}x \Big|_0^L} - \frac{L}{n\pi}\int_0^L f'' \sin\frac{n\pi}{L}x \, dx \right]$$

$$= -\frac{2L}{\pi^2 n^2}\int_0^L f'' \sin\frac{n\pi}{L}x \, dx$$

$$|A_n| \leq \frac{2L}{\pi^2 n^2} M_1 \Rightarrow n^2 |A_n| \leq \frac{2L}{\pi^2} M_1$$

where $M_1 = \int_0^L |f''| \, dx$

f'' is continuous in $[0, L]$ or

$|f''|$ is integrable in $(0, L)$.

$$B_n = \frac{2}{n\pi c}\left[-\frac{L}{n\pi} g(x)\cos\frac{n\pi}{L}x\Big|_0^L + \frac{L}{n\pi}\int_0^L g'(x)\cos\frac{n\pi}{L}x\,dx \right]$$

$$= \frac{2L}{\pi^2 c n^2}\int_0^L g'(x)\cos\frac{n\pi}{L}x\,dx$$

$$n^2|B_n| \le \frac{2L}{\pi^2 c} M_4 \quad , \quad M_4 = \int_0^L |g'|\,dx$$

hence g' is continuous in $[0,L)$.

$g(0)=g(L)=0$

For the existence of the solution

i) $f(0)=f(L)=0$ and f has a continuous second derivative in $[0,L]$

ii) $g(0)=g(L)=0$ and g has a continuous first derivative in $[0,L]$

~~there exists~~ The solution of the initial and boundary value problem exists. With the above conditions $u(x,t)$ satisfies the PDE, BCs and the IC.

<u>Uniqueness</u>: Assume the contrary. ~~Assume that there~~ ~~there exist~~ ~~solution obtained u is for two different BCs. and ICs.~~
Assume that there are two different solutions u_1 and u_2 for the same IC and BCs (same data)

Define $u(x,t) = u_1(x,t) - u_2(x,t)$

PDE : $u_{tt} = c^2 u_{xx}$, $0 < x < L, t > 0$

BCs : $u(0,t) = u(L,t) = 0$

ICs : $u(x,0) = 0$, $u_t(x,0) = 0$

zero data.

Lemma: The wave equation with zero data has only the trivial solution.

proof: Define the energy functional

$$E(t) = \int_0^L [u_t^2 + c^2 u_x^2] dx$$

$E(0) = 0$, $E(t) \geq 0$, $t > 0$

$$\frac{dE}{dt}(t) = \int_0^L [2 u_t u_{tt} + c^2 u_x u_{xt}] dx$$

$$= 2\int_0^L [c^2 u_t u_{xx} + u_x u_{xt}] dx$$

$$= 2c^2 \int_0^L (u_t u_x)_x dx = 2c^2 \, u_t u_x \Big|_0^L = 0$$

hence $E(t) = C$, $\forall t$

but $E(0) = 0 \Rightarrow C = 0$

$E(t) = 0$ $\forall t$.

$\Rightarrow u_t = u_x = 0$ $\forall x, \forall t$

$\Rightarrow u = \text{const.}$ but this constant is zero

$u(x,t) = 0 \Rightarrow u_1(x,t) = u_2(x,t)$ contradiction. proves uniqueness.

c) stability. let

$$f_1, g_1 \rightarrow u_1$$

$$f_2, g_2 \rightarrow u_2$$

$$u_1(x,t) - u_2(x,t) = \sum_{n=1}^{\infty}{}' \sin\left(\frac{n\pi}{L}x\right) \left[(A_n^1 - A_n^2) \cos\frac{n\pi}{L}ct + (B_n^1 - B_n^2) \sin\frac{n\pi}{L}ct \right]$$

$$|u_1 - u_2| \leq \sum_{n=1}^{\infty}{}' \left(|A_n^1 - A_n^2| + |B_n^1 - B_n^2| \right)$$

$$|A_n^1 - A_n^2| \leq \frac{2L}{\pi^2 n^2} M_{12} \quad , \quad M_{12} = \int^L |f_1'' - f_2''| \, dx.$$

$$|B_1^1 - B_2^2| \leq \frac{2L}{\pi^2 n^2} N_{12} \, , \quad N_{12} = \int_0^L |g_1' - g_2'| \, dx.$$

$$\Rightarrow \quad |u_1 - u_2| \leq \frac{2L}{\pi^2} (M_{12} + N_{12}) \frac{\pi^2}{6} = \frac{2L}{6} (M_{12} + N_{12})$$

$$\Rightarrow \quad |u_1 - u_2| \leq \frac{L}{3} \left(\| f_1 - f_2 \|_2 + \| g_1 - g_2 \|_1 \right.$$

$$\| f_1 - f_2 \|_2 = \max \left\{ \cancel{|f_1'' - f_2''|} \; |f_1 - f_2| , |f_1' - f_2'|' , |f_1'' - f_2''| \right\} < \varepsilon/2$$

$$\| g_1 - g_2 \|_1 = \max \left\{ |g_1 - g_2| , |g_1' - g_2'| \right\} < \varepsilon/2$$

$$\Rightarrow \quad \boxed{|u_1 - u_2| \leq \frac{L}{3} \varepsilon}$$

NOTE

Stability of the wave equation with fixed ends

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{L}\left[-\frac{L}{n\pi}\overset{0}{\cancel{f(x)}}\cos\frac{n\pi}{L}x\Big|_0^L + \frac{L}{n\pi}\int_0^L f'\cos\frac{n\pi}{L}x\,dx\right]$$

$$= \frac{2}{n\pi}\left[\frac{L}{n\pi}f'(x)\sin\frac{n\pi}{L}x\Big|_0^L - \frac{L}{n\pi}\int_0^L f''\sin\frac{n\pi}{L}x\,dx\right]$$

$$= -\frac{2L}{(n\pi)^2}\int_0^L f''\sin\frac{n\pi}{L}x\,dx$$

$$\frac{n\pi}{L}c\,B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{n\pi}\int_0^L g'\cos\frac{n\pi}{L}x\,dx$$

$$B_n = \frac{2L}{(n\pi)^2}\int_0^L g'\cos\frac{n\pi}{L}x\,dx$$

$$|u_1-u_2| \le \cancel{\sum(|A_n|+|B}\ \sum\left(|A_n^1-A_n^2| + |B_n^1-B_n^2|\right)$$

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$$A_n^1 - A_n^2 = -\frac{2L}{(n\pi)^2}\int_0^L \{f_1'' - f_2''\} \sin\frac{n\pi}{L}x \, dx$$

$$|A_n^1 - A_n^2| \leq \frac{2L}{(n\pi)^2} M_{12}$$

$$M_{12} = \int_0^L |f_1'' - f_2''| \, dx = \max|f_1'' - f_2''| \, L$$

$$|A_n^1 - A_n^2| \leq \frac{2L^2}{(n\pi)^2} \max|f_1'' - f_2''|$$

$$|B_n^1 - B_n^2| \leq \frac{2L^2}{(n\pi)^2} \max|g_1' - g_2'|$$

let $\|f_1 - f_2\|_2 = \max\{\max|f_1 - f_2|, \max|f_1' - f_2'|, \max|f_1'' - f_2''|\}$

$< \varepsilon$

$$\|g_1 - g_2\|_1 = \max\{\max|g_1 - g_2|, \overset{\max}{}|g_1' - g_2'|\} < \varepsilon$$

$$|u_1 - u_2\| \leq \frac{2L}{\pi^2}\left(\|f_1 - f_2\|_2 + \|g_1 - g_2\|_1\right)\frac{\pi^2}{6}$$

$$\leq \frac{L}{3} 2\varepsilon$$

$$|u_1 - u_2| \leq \frac{2L}{3}\varepsilon$$

<u>Stable.</u>

Wave Equation: Vibrating string with infinite Length.

PDE : $u_{tt} = c^2 u_{xx}$, $x \in \mathbb{R}$, $t > 0$

IC : $$\left.\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array}\right\} x \in \mathbb{R}$$

Solution:

$$u_{tt} - c^2 u_{xx} = (D^2 - c^2 D'^2) u = 0$$

$$(D_t - c D'_x)(D_t + c D'_x) u = 0$$

$$\Rightarrow \quad u(x,t) = F(x-ct) + G(x+ct).$$

Solutions like these called the "travelling waves with speed c in the $+$ and $-$ x-directions. Here F and G are arbitrary functions.

Initial conditions

$$u(x,0) = F(x) + G(x) = f(x)$$

$$u_t(x,t) = -c\, F_\xi + c\, G_\eta \quad , \ \xi = x - ct , \ \eta = x + ct$$

$$u_t(x,0) = -c\, F_x(x) + c\, G_x(x) = g(x)$$

$$-c\, F(x) + c\, G(x) = \int^x g(x)\, dx$$

$$F(x) + G(x) = f(x)$$

$$-F(x) + G(x) = \frac{1}{c}\int^{x} g(x)\,dx.$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int^{x} g(\xi)\,d\xi$$

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int^{x} g(\xi)\,d\xi.$$

Hence

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$= \frac{1}{2} f(x-ct) + \frac{1}{2c}\int^{x-ct} g(\xi)\,d\xi$$

$$+ \frac{1}{2} f(x+ct) - \frac{1}{2c}\int^{x+ct} g(\xi)\,d\xi$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$

Given f C^2. and g (integrable) (twice differentiable functions solution exists.

<u>stability</u>

$f_1, g_1 \to u_1$

$f_2, g_2 \to u_2$

$$u_1 - u_2 = \frac{1}{2}\left[f_1(x+ct) - f_2(x+ct) + f_1(x-ct) - f_2(x-ct)\right]$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} (g_1 - g_2)\,d\xi$$

Stability of the solutions

$(f_1, g_1) \rightarrow u_1(x,t)$

$(f_2, g_2) \rightarrow u_2(x,t)$

$$u_1(x,t) - u_2(x,t) = \frac{1}{2}\left[f_1(x+ct) - f_2(x+ct) \right]$$
$$+ \frac{1}{2}\left[f_1(x-ct) - f_2(x-ct) \right]$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} \left(g_1(\xi,t) - g_2(\xi,t) \right) d\xi$$

$$|u_1 - u_2| \leq \frac{1}{2}\left| f_1(x+\cancel{\varepsilon}ct) - f_2(\cancel{\alpha}x+ct)\right| + \frac{1}{2}\left| f_1(x-ct) - f_2(x-ct)\right|,$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} \left| g_1(\xi,\cancel{t}) - g_2(\xi\cancel{t})\right| d\xi$$

let $|f_1 - f_2| < \varepsilon$, $|g_1 - g_2| < \varepsilon$ $\forall (x,t)$

$\forall x \in \mathbb{R}$

$\forall t \in [0,T]$

$$|u_1 - u_2| \leq \varepsilon + \frac{\varepsilon}{2c}\int_{x-ct}^{x+ct} d\xi = \varepsilon\left(1 + \frac{1}{2c}\,2ct\right)$$

$$\leq \varepsilon(1+T)$$ since $0 < T < \infty$

Solution is stable

$$|u_1 - u_2| \leq \varepsilon + \frac{1}{2c} \varepsilon 2cT \leq \varepsilon(1+T)$$

for $t \in [0,T]$.

Chapter 5. Sturm-Liouville Eigenvalue Problems

In our previous problems we studied an eigenvalue problem

$$\phi'' + \lambda \phi = 0 , \quad \phi(0) = 0 , \ \phi(L) = 0$$

- Eigenvalues are real and infinite number, λ_n
- for each eigenvalue we have an eigenfunction ϕ_n.
- Eigenfunctions with different eigenvalues are orthogonal.

From which problem we have "Eigenvalue Problems".

i) Heat flow in a nonuniform rod

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x} \right) + \varphi$$

$\varphi(x,t)$ source

c, ρ, K_0 depend on x in general.

Let $\varphi = \alpha u$, α depend on x.

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x} \right) + \alpha u$$

Separation of variables

$$u(x,t) = \phi(x,t)\, h.$$

$$\frac{1}{h}\frac{dh}{dt} = \frac{1}{c\rho\phi}\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + \frac{\alpha}{c\rho} = -\lambda$$

λ is the separation constant

i) $h' + \lambda h = 0$

ii) $\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\phi = 0$

i) has exponentially decreasing ($\lambda > 0$) and exponentially increasing and steady solution for $\lambda \leq 0$. It may physically possible to have exponentially increasing and steady solutions because there exists a source term α. If $\alpha > 0$ we can have such solutions.

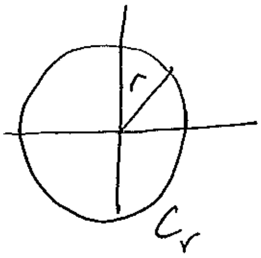
Circularly symmetric Heat flow

In two dimensional heat flow problem

$$u_t = k \nabla^2 u$$

$$= k\left[\frac{1}{r}\left(r \frac{\partial u}{\partial r} \right)_r + \frac{1}{r^2} u_{\theta\theta} \right]$$ in polar coordinates.

Circulary symmetric solutions



at any point on C_r $u(x,t)$ has the same value. This means that $u(r,\theta,t)$

does not depend on θ. Hence

$$u_t = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right)$$

Separation of variables: $u(r,t) = \phi(r)\, h(t)$

$$h' + \lambda h$$

$$\frac{d}{dr}\left(r \frac{d\phi}{dr} \right) + \lambda r \phi = 0$$ "an eigenvalue problem"

A different type eigenvalue problem. $\lambda < 0$ not possible. We have singularity conditions.

$|u(0,t)| < \infty$ and homogeneous conds.

5.3 Sturm Liouville Eigenvalue Problem.

General Classification

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right)+q\phi+\lambda\sigma\phi=0\,,\qquad (*)$$

$a<x<b$, λ is the eigenvalue

Simple cases

i) $\phi''+\lambda\phi=0$ $\qquad p=1, q=0, \sigma=1$

ii) $(K_0\phi')'+\alpha\phi+\lambda c\rho\phi=0$, $\qquad p=K_0, q=\alpha$
$\sigma=c\rho$

iii) Vibrations of nonuniform string

$T_0\,\phi''+\alpha\phi+\lambda\rho_0\phi=0$ $\quad$, $p=T_0$ const
$q=\alpha, \sigma=\rho_0$

iv) Circularly symmetric heat flow

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)+\lambda r\phi=0$$

$x=r, p(x)=x$
$q(x)=0, \sigma(x)=x$

Equation $(*)$ is known as a "Sturm-Liouville" differential equation

~~Equation (*) is known as a Sturm-Liouville differential equation~~

Boundary conditions

The linear homogeneous boundary conditions that we have studied are of the form to follow.

BCs	Heat flow	Vibrating String	Mathematical terminology
$\phi = 0$	Fixed (zero) temperature	Fixed (zero) displacement (end point)	Dirichlet BCs.
$\frac{d\phi}{dt} = 0$	Insulated	Free ends	Neuman Condi
$\frac{d\phi}{dx} = \pm h\phi$	Homogeneous Newton's law of cooling	Homogeneous elastic BCs	Robin Condit
$\phi(-L) = \phi(L)$ $\phi'(-L) = \phi(L)$	Perfect termal contact		Periodic BCs
$\lvert\phi(0)\rvert < \infty$	Bounded temperature		Singularity condition

Regular Sturm-Liouville Eigenvalue Problem

A regular Sturm-Liouville eigenvalue problem consists of the Sturm-Liouville DE

DE: $$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)+q(x)\phi+\lambda\sigma(x)\phi=0$$

where $a<x<b$. Subject to the BCs

BCs: $$\beta_1\phi(a)+\beta_2\frac{d\phi}{dx}(a)=0$$
$$\beta_3\phi(b)+\beta_2\frac{d\phi}{dx}(b)=0$$

where $\beta_1, \beta_2, \beta_3$, and β_4 are ~~and $p>0, \sigma>0$~~ real constants. In addition, the coefficients p, q, σ must be real and continuous everywhere in $[a,b]$ (including the end points) and $p>0$, $\sigma>0$ $\forall x\in[a,b]$.

For the regular Sturm-Liouville problems many important general teorems exist.

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subject to the boundary conditions that we have discussed (excluding periodic and singular cases):

$$\boxed{\begin{aligned} \beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0 \\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0, \end{aligned}} \tag{5.3.3}$$

where β_i are real. In addition, to be called regular, the coefficients $p, q,$ and σ must be real and continuous everywhere (including the end points) and $p > 0$ and $\sigma > 0$ everywhere (also including the endpoints). For the regular Sturm-Liouville eigenvalue problem, many important general theorems exist. In Sec. 5.5 we will prove these results, and in Secs. 5.7 and 5.8 we will develop some more interesting examples that illustrate the significance of the general theorems.

Statement of theorems. At first let us just state (in one place) all the theorems we will discuss more fully later (and in some cases prove). For any regular Sturm-Liouville problem, all of the following theorems are valid:

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$
   a. There is a smallest eigenvalue, usually denoted λ_1.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$
Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient:**
$$\lambda = \frac{-p\phi\, d\phi/dx|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2\sigma\,dx},$$
where the boundary conditions may somewhat simplify this expression.

It should be mentioned that for Sturm-Liouville eigenvalue problems that are not "regular," these theorems maybe valid. An example of this is illustrated in Secs. 7.7 and 7.8.

5.3.3 Example and Illustration of Theorems

We will individually illustrate the meaning of these theorems (before proving many of them in Sec. 5.5) by referring to the simplest example of a regular Sturm-Liouville problem:

$$\boxed{\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned}} \tag{5.3.4}$$

The constant-coefficient differential equation has zero boundary conditions at both ends. As we already know, the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{with} \quad \phi_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots,$$

giving rise to a Fourier sine series.

1. **Real eigenvalues.** Our theorem claims that all eigenvalues λ of a regular Sturm-Liouville problem are real. Thus, the eigenvalues of (5.3.4) should all be real. We know that the eigenvalues are $(n\pi/L)^2, n = 1, 2, \ldots$. However, in determining this result (see Sec. 2.3.4) we analyzed three cases: $\lambda > 0, \lambda = 0$, and $\lambda < 0$. We did not bother to look for complex eigenvalues because it is a relatively difficult task and we would have obtained no additional eigenvalues other than $(n\pi/L)^2$. This theorem (see Sec 5.5 for its proof) is thus very useful. It guarantees that we do not even have to consider λ being complex.

2. **Ordering of eigenvalues.** There is an infinite number of eigenvalues for (5.3.4), namely $\lambda = (n\pi/L)^2$ for $n = 1, 2, 3, \ldots$. Sometimes we use the notation $\lambda_n = (n\pi/L)^2$. Note that there is a smallest eigenvalue, $\lambda_1 = (\pi/L)^2$, but no largest eigenvalue since $\lambda_n \to \infty$ as $n \to \infty$. Our theorem claims that this idea is valid for any regular Sturm-Liouville problem.

3. **Zeros of eigenfunctions.** For the eigenvalues of (5.3.4), $\lambda_n = (n\pi/L)^2$, the eigenfunctions are known to be $\sin n\pi x/L$. We use the notation $\phi_n(x) = \sin n\pi x/L$. The eigenfunction is unique (to within an arbitrary multiplicative constant).

An important and interesting aspect of this theorem is that we claim that for all regular Sturm-Liouville problems, the nth eigenfunction has exactly $(n-1)$ zeros, not counting the endpoints. The eigenfunction ϕ_1 corresponding to the smallest eigenvalue $(\lambda_1, n = 1)$ should have no zeros in the interior. The eigenfunction ϕ_2 corresponding to the next smallest eigenvalue $(\lambda_2, n = 2)$ should have exactly one zero in the interior; and so on. We use our eigenvalue problem (5.3.4) to illustrate these properties. The eigenfunctions $\phi_n(x) = \sin n\pi x/L$ are sketched in Fig. 5.3.1 for $n = 1, 2, 3$. Note that the theorem is verified (since we only count zeros at interior

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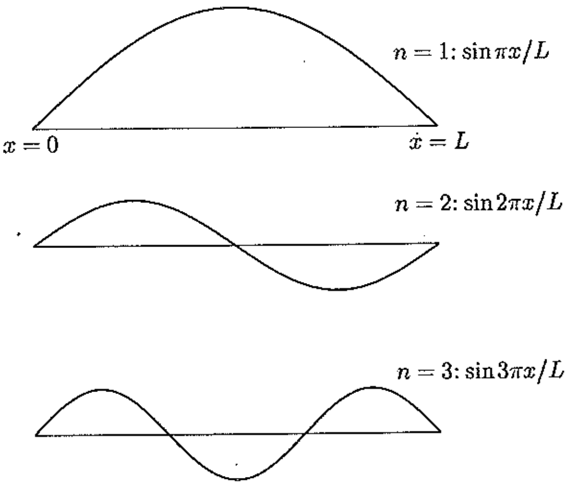

Figure 5.3.1 Zeros of eigenfunctions $\sin n\pi x/L$.

points); $\sin \pi x/L$ has no zeros between $x = 0$ and $x = L$, $\sin 2\pi x/L$ has one zero between $x = 0$ and $x = L$, and $\sin 3\pi x/L$ has two zeros between $x = 0$ and $x = L$.

4. **Series of eigenfunctions.** According to this theorem, the eigenfunctions can always be used to represent any piecewise smooth function $f(x)$,

$$\boxed{f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).} \tag{5.3.5}$$

Thus, for our example (5.3.4),

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}.$$

We recognize this as a Fourier sine series. We know that any piecewise smooth function can be represented by a Fourier sine series and the infinite series converges to $[f(x+)+f(x-)]/2$ for $0 < x < L$. It converges to $f(x)$ for $0 < x < L$, if $f(x)$ is continuous there. This theorem thus claims that the convergence properties of Fourier sine series are valid for all series of eigenfunctions of any regular Sturm-Liouville eigenvalue problem. Equation (5.3.5) is refereed to as an expansion of $f(x)$ in terms of the eigenfunctions $\phi_n(x)$ or, more simply, as an eigenfunction expansion. It is also called a **generalized Fourier series of $f(x)$**. The coefficients a_n are called the coefficients of the eigenfunction expansion or the **generalized Fourier coefficients.** The fact that rather arbitrary functions may be represented in terms of an infinite series of eigenfunctions will enable us to solve partial differential equations by the method of separation of variables.

5. **Orthogonality of eigenfunctions.** The preceding theorem enables a function to be represented by a series of eigenfunctions, (5.3.5). Here we will show how to determine the generalized Fourier coefficients, a_n. According to the important theorem we are now describing, the eigenfunctions of any regular Sturm-Liouville eigenvalue problem will always be orthogonal. The theorem states that a weight $\sigma(x)$ must be introduced into the orthogonality relation:

$$\boxed{\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\ dx = 0,} \quad \text{if} \quad \lambda_n \neq \lambda_m. \tag{5.3.6}$$

Here $\sigma(x)$ is the possibly variable coefficient that multiplies the eigenvalue λ in the differential equation defining the eigenvalue problem. Since corresponding to each eigenvalue there is only one eigenfunction, the statement "if $\lambda_n \neq \lambda_m$" in (5.3.6) may be replaced by "if $n \neq m$." For the Fourier sine series example, the defining differential equation is $d^2\phi/dx^2 + \lambda\phi = 0$, and hence a comparison with the form of the general Sturm-Liouville problem shows that $\sigma(x) = 1$. Thus, in this case the weight is 1, and the orthogonality condition, $\int_0^L \sin n\pi x/L \sin m\pi x/L\, dx = 0$, follows if $n \neq m$, as we already know.

As with Fourier sine series, we use the orthogonality condition to determine the generalized Fourier coefficients. In order to utilize the orthogonality condition (5.3.6), we must multiply (5.3.5) by $\phi_m(x)$ *and* $\sigma(x)$. Thus,

$$f(x)\phi_m(x)\sigma(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)\phi_m(x)\sigma(x),$$

where we assume these operations on infinite series are valid, and hence introduce equal signs. Integrating from $x = a$ to $x = b$ yields

$$\int_a^b f(x)\phi_m(x)\sigma(x)\ dx = \sum_{n=1}^{\infty} a_n \int_a^b \phi_n(x)\phi_m(x)\sigma(x)\ dx.$$

Since the eigenfunctions are orthogonal [with weight $\sigma(x)$], all the integrals on the right-hand side vanish except when n reaches m:

$$\int_a^b f(x)\phi_m(x)\sigma(x)\ dx = a_m \int_a^b \phi_m^2(x)\sigma(x)\ dx.$$

The integral on the right is nonzero since the weight $\sigma(x)$ must be positive (from the definition of a regular Sturm-Liouville problem), and hence we may divide by it to determine the generalized Fourier coefficient a_m:

$$\boxed{a_m = \frac{\displaystyle\int_a^b f(x)\phi_m(x)\sigma(x)\ dx}{\displaystyle\int_a^b \phi_m^2(x)\sigma(x)\ dx}.} \tag{5.3.7}$$

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In the example of a Fourier sine series, $a = 0, b = L, \phi_n = \sin n\pi x/L$ and $\sigma(x) = 1$. Thus, if we recall the known integral that $\int_0^L \sin^2 n\pi x/L \; dx = L/2$, (5.3.7) reduces to the well-known formula for the coefficients of the Fourier sine series. It is not always possible to evaluate the integral in the denominator of (5.3.7) in a simple way.

6. **Rayleigh quotient.** In Sec. 5.6 we will prove that the eigenvalue may be related to its eigenfunction in the following way:

$$\lambda = \frac{-p\phi \; d\phi/dx\big|_a^b + \displaystyle\int_a^b [p(d\phi/dx)^2 - q\phi^2] \; dx}{\int_a^b \phi^2 \sigma \; dx}, \tag{5.3.8}$$

known as the **Rayleigh quotient.** The numerator contains integrated terms and terms evaluated at the boundaries. Since the eigenfunctions cannot be determined without knowing the eigenvalues, this expression is never used directly to determine the eigenvalues. However, interesting and significant results can be obtained from the Rayleigh quotient without solving the differential equation. Consider the Fourier sine series example (5.3.4) that we have been analyzing: $a = 0, b = L, p(x) = 1, q(x) = 0$, and $\sigma(x) = 1$. Since $\phi(0) = 0$ and $\phi(L) = 0$, the Rayleigh quotient implies that

$$\lambda = \frac{\int_0^L (d\phi/dx)^2 dx}{\int_0^L \phi^2 \; dx}. \tag{5.3.9}$$

Although this does not determine λ since ϕ is unknown, it gives useful information. Both the numerator and the denominator are ≥ 0. Since ϕ cannot be identically zero and be called an eigenfunction, the denominator cannot be zero. Thus, $\lambda \geq 0$ follows from (5.3.9). Without solving the differential equation, we immediately conclude that there cannot be any negative eigenvalues. When we first determined eigenvalues for this problem, we worked rather hard to show that there were no negative eigenvalues (see Sec. 2.3). Now we can simply apply the Rayleigh quotient to eliminate the possibility of negative eigenvalues *for this example.* Sometimes, as we shall see later, we can also show that $\lambda \geq 0$ in harder problems.

Furthermore, even the possibility of $\lambda = 0$ can sometimes be analyzed using the Rayleigh quotient. For the simple problem (5.3.4) with zero boundary conditions, $\phi(0) = 0$ and $\phi(L) = 0$, let us see if it is possible for $\lambda = 0$ directly from (5.3.9). $\lambda = 0$ only if $d\phi/dx = 0$ for all x. Then, by integration, ϕ must be a constant for all x. However, from the boundary conditions [either $\phi(0) = 0$ or $\phi(L) = 0$], that constant must be zero. Thus, $\lambda = 0$ only if $\phi = 0$ everywhere. But if $\phi = 0$ everywhere, we do not call ϕ an eigenfunction. Thus, $\lambda = 0$ is not an eigenvalue *in this case,* and we have further concluded that $\lambda > 0$; all the eigenvalues must be positive. This is concluded *without* using solutions of the differential equation. The known eigenvalues in this example, $\lambda_n = (n\pi/L)^2, n = 1, 2, \ldots$, are clearly consistent with the conclusions from the Rayleigh quotient. Other applications of the Rayleigh quotient will appear in later sections.

5.5. Self-Adjoint Operators and Sturm-Liouville Eigenvalue Problem

We shall prove some of the properties of regular sturm-Liouville eigenvalue problem

DE : $$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0$$

$$a < x < b$$

BCs $$B_1(u) \equiv \beta_1 \phi(a) + \beta_2 \phi'(a) = 0$$
$$B_2(u) \equiv \beta_3 \phi(b) + \beta_4 \phi'(a) = 0$$

β_i are real p, q, σ are real continuous functions and $p(x) > 0$, $\sigma(x) > 0$ $\forall x \in [a,b]$.

The following three problems (properties) are important but we will not prove them

i) There are infinitely many eigenvalues

ii) Eigenfunctions form a "complete set" form a basis of square integrable function space. In an another saying that Any piecewise smooth functions can be expanded interms of the eigenfunctions of the regular Sturm-Liouville problem.
L^2 space is complete

iii) ϕ_n 's has $n-1$ zeros in (a,b).

- Properties of regular Sturm-Liouville problem

Sturm-Liouville operator

$$L = \frac{d}{dx} p \frac{d}{dx} + q(x).$$

$$L : \text{functions} \longrightarrow \text{functions}$$

$$L(y) = \frac{d}{dx}\left(p \frac{dy}{dx}\right) + q(x) y$$

Sturm-Liouville DE is shortly given by

$$L(\phi) + \lambda \sigma(x) \phi(x) = 0$$

<u>Lagrange Identity</u>: let u and v be two functions, then

$$L(u) = \frac{d}{dx}\left(p \frac{du}{dx}\right) + q u, \quad L(v) = \frac{d}{dx}\left(p \frac{dv}{dx}\right) + q v$$

Hence

$$\begin{aligned} u L(v) - v L(u) &= u \frac{d}{dx}\left(p \frac{dv}{dx}\right) + q \cancel{uv} \\ &\quad - v \frac{d}{dx}\left(p \frac{du}{dx}\right) - q \cancel{vu} \\ &= \frac{d}{dx}\left[p \left(u \frac{dv}{dx} - v \frac{du}{dx} \right)\right] \end{aligned}$$

Hence we have

$$uL(v) - vL(u) = \frac{d}{dx}\left[p(uv' - vu') \right]$$

This is the Lagrange identity (differential form)

Green's Formula : Integrating the Lagrange identity in (a,b) we get Green's Formula

$$\int_a^b \left[uL(v) - vL(u) \right] dx = p(uv' - vu')\Big|_a^b$$

with some additional conditions (namely the BCs) if the surface term vanishes.

$$p(uv' - vu')\Big|_a^b = 0$$

then we get

$$\int_a^b \left[uL(v) - vL(u) \right] dx = 0$$

For example suppose that u, and v are any two functions that satisfy the following BCs

$$\phi(a) = 0$$

$$\phi'(b) + h\,\phi(b) = 0$$

This means that

$$u(a)=0, \quad v(a)=0$$
$$u'(b)+h\,u(a)=0, \quad v'(b)+h\,v(b)=0$$

satisfying the same BCs.

$$p(uv'-vu')\Big|_a^b = p(b)\left[u(b)v'(b)-v(b)u'(b)\right]$$
$$-\,p(a)\left[\underset{0}{\cancel{u(a)}}\,v'(a)-\underset{0}{\cancel{v(a)}}\,u'(a)\right]$$
$$= p(b)\left[-h\,u(b)v(b)+h\,v(b)u(b)\right]=0$$
$$=0 \quad \checkmark$$

Hence for such functions

$$\int_a^b \left[u\,L(v)-v\,L(u)\right]dx=0$$

Boundary term vanishes also for the BCs.

$$\beta_1\,\phi(a)+\beta_2\,\phi'(a)=0$$
$$\beta_3\,\phi(b)+\beta_4\,\phi'(b)=0$$

satisfied by u and v (the same constants β_i).

Hence, if u and v are any two functions satisfying the same set of homogeneous BCs (of the regular Sturm-Liouville type), then

$$\int_a^b [u L(v) - v L(u)]\, dx = 0$$

such operators are called "self-adjoint".

Remark: Green's formula is also valid for generalized periodic BCs. like

$$\phi(a) = \phi(b), \quad \text{and} \quad p(a)\phi'(a) = p(b)\phi'(b).$$

Properties of Sturm-Liouville Eigenvalue problem.

1) Orthogonal eigenfunctions: Eigenfunctions corresponding to different eigenvalues are orthogonal

$$L(\phi_n) + \lambda_n \sigma(x) \phi_n = 0 \qquad \lambda_n \neq \lambda_m$$
$$L(\phi_m) + \lambda_m \sigma(x) \phi_m = 0$$

Green's identity

$$\int_a^b [\phi_m L(\phi_n) - \phi_n L(\phi_m)]\, dx = p(x)\left(\phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx}\right)\Bigg|_a^b$$

For many different BCs, regular S-L type, periodic case and also singular case The RHS the surface term is zero

Hence

$$\int_a^b [\phi_m L(\phi_n) - \phi_n L(\phi_m)] dx = 0$$

$$\int_a^b (\lambda_n - \lambda_m)\, \sigma(x)\, \phi_n(x)\, \phi_m(x)\, dx = 0$$

$$(\lambda_n - \lambda_m) \int_a^b \sigma(x)\, \phi_n(x)\, \phi_m(x)\, dx = 0$$

$$\lambda_n \neq \lambda_m \Rightarrow \int_a^b \sigma(x)\, \phi_n(x)\, \phi_m(x)\, dx = 0$$

Innerproduct $\langle \phi_n, \phi_m \rangle = \int_a^b \sigma(x)\, \phi_n(x)\, \phi_m(x)\, dx$

where $\sigma(x) > 0$, $\forall x \in [a,b]$ called the weight function. Orthogonality

$$\langle \phi_n, \phi_m \rangle = 0 \quad \text{for all } m \neq n.$$

2) Real eigenvalues. Since $p(x)$ and $\sigma(x)$ are real $\bar{L} = L$ real ~~[illegible]~~ operator

$$L(\phi) + \lambda \sigma \phi = 0$$

$$\overline{L(\phi)} + \bar{\lambda} \sigma \bar{\phi} = 0 \Rightarrow L(\bar{\phi}) + \bar{\lambda} \sigma \bar{\phi}$$

$$\phi L(\bar{\phi}) - \bar{\phi} L(\phi) = \frac{d}{dx}\left(p(x)(\phi \bar{\phi}' - \bar{\phi} \phi')\right)$$

$$\int_a^b [\phi L(\bar{\phi}) - \bar{\phi} L(\phi)] dx = 0 \quad \text{with BCs.}$$

$$\Rightarrow \int_a^b \sigma (\lambda - \bar{\lambda}) \phi \bar{\phi}\, dx = (\lambda - \bar{\lambda}) \int \sigma \phi \bar{\phi}\, dx = 0$$

$$\lambda = \bar{\lambda}$$

3) For given eigenvalues we have unique eigenfunctions. for regular S-L problem

Let us assume that there are two eigenfunctions ϕ_1, and ϕ_2 for a given eigenvalue λ

$$L(\phi_1) + \lambda\sigma\phi_1 = 0$$
$$L(\phi_2) + \lambda\sigma\phi_2 = 0$$

$$\Rightarrow \quad \phi_2 L(\phi_1) - \phi_1 L(\phi_2) = 0$$

$$\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = \frac{d}{dx}\left[p(\phi_2\phi_1' - \phi_1\phi_2')\right] = 0$$

$$\Rightarrow \quad p(\phi_2\phi_1' - \phi_1\phi_2') = C \quad (\text{constant}) \quad \forall x$$

for regular SL problem $C = 0$. For instance let

$$\phi'(a) + h\phi(a) = 0 \quad \Rightarrow$$

$$p(a)\left(-\phi_2(a)\, h\,\phi_1(a) + \phi_1(a)\, h\,\phi_2(a)\right) = 0 = C.$$

$$\Rightarrow \quad \phi_2\phi_1' - \phi_1\phi_2' = 0 \quad \forall x$$

$$\text{or} \qquad \phi_2(x) = c\,\phi_1(x)$$

This shows that two eigenfunctions ϕ_1 and ϕ_2 corresponding to the same eigenvalue must be integral multiple of each other. Hence Eigenfunction are unique.

Remark: For periodic BCs there corresponds two different eigenfunctions. to a given eigenvalue. Heat conduction on a circular ring. is an example.

$$\lambda_0 = 0 \qquad \phi = 1$$

$$\lambda = n^2 \Rightarrow (\sin n\theta, \cos n\theta)$$

5.6 Rayleigh Quotient

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0$$

multiplying by ϕ

$$\int_a^b \left[\phi\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi^2\right]dx$$
$$+ \lambda\int_a^b \sigma(x)\,\phi^2(x)\,dx = 0$$

or

$$p(x)\,\phi\frac{d\phi}{dx}\Big|_a^b + \cancel{p(x)\left(\frac{d\phi}{dx}\right)^2\Big|_a^b}$$
$$+\int_a^b\left[-p(x)\left(\frac{d\phi}{dx}\right)^2 + q\phi^2\right]dx$$
$$+\lambda\int_a^b \sigma(x)\,\phi^2(x)\,dx = 0$$

$$\lambda = \frac{1}{\|\phi\|^2}\left[-p(x)\,\phi\frac{d\phi}{dx}\Big|_a^b + \int_a^b\left[p(x)\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx\right]$$

if a) $p\phi\phi'\Big|_a^b = 0$ by the BCs

b) $q(x) \leq 0 \quad \forall x \in (a,b)$

$\Rightarrow \lambda \geq 0$

The minimum value of the Rayleigh quotient for all continuous functions satisfying the BCs is the lowest eigenvalue

$$\lambda_1 = \min \frac{-p u u' \Big|_a^b + \int_a^b [p u'^2 - q u^2] dx}{\int_a^b u^2 \sigma dx}$$

where λ_1 represents the smallest eigenvalue

The minimization includes all continuous functions that satisfy the BCs. The minimum is obtained only for $u = \phi_1(x)$ the lowest eigenfunctions.

Trial function u_T for the heat equation
let u_T be any function satisfying the BCs.

$$\lambda_1 \leq RQ[u_T] = \frac{-p u_T u_T' \Big|_a^b + \int_a^b [p u_{T_x}^2 - q u^2] dx}{\int_a^b u_T^2 \sigma dx}$$

Ex: $\phi'' + \lambda \phi = 0$

$\phi(0) = 0, \quad \phi(1) = 0, \quad \lambda_n = n^2\pi^2$

$\lambda_1 = \pi^2$

$$\pi^2 \leq \frac{\int_0^1 \left(\frac{du_T}{dx}\right)^2 dx}{\int_0^1 u_T^2\, dx}$$

$u_T(0) = 0$
$u_T(1) = 0$

u_T may not satisfy the heat eqn.

Example

$$u_T = \begin{cases} x & x < \frac{1}{2} \\ 1-x & x > \frac{1}{2} \end{cases}$$

$u_T(0) = 0$
$u_T(1) = 0$

$$\int_0^1 \left(\frac{du_T}{dx}\right)^2 dx = \int_0^{1/2} dx + \int_{1/2}^1 dx = 1.$$

$$\int_0^1 \phi^2 dx = \int_0^{1/2} x^2 dx + \int_{1/2}^1 (1-x)^2 dx$$

$$= \frac{1}{3}\ \frac{1}{8} + (\cancel{1} - 1/2) - (\cancel{1} - \frac{1}{4}) + \frac{1}{3}\left(1 - \frac{1}{8}\right)$$

$$= \frac{1}{24} - \frac{1}{4} + \frac{7}{24} = -\frac{5}{\underset{24}{\cancel{6}}} + \frac{7}{24} = \frac{2}{24} = \frac{1}{12}.$$

$R\varphi[U_T] = 12$ $\pi^2 \leq 12$ ✓

$U_T = x - x^2$, $R\varphi(U_T) = 10$ $\pi^2 \leq 10$

Proof:

$$R\varphi[u] = \frac{-\int_a^b u L(u)\,dx}{\int_a^b u^2 \sigma\, d\sigma}$$

$$\left\{ \quad L(u) + \lambda \sigma u = 0 \qquad u L(u) + \lambda \sigma u^2 = 0\right.$$

$$\int_a^b u L(u)\,dx + \lambda \|u\|^2 = 0 \quad \Rightarrow \lambda = \frac{-1}{\|u\|^2} \int_a^b u L(u)\,dx.$$

We expand u in terms of eigenfunctions Φ_n

$$u = \sum_{n=1}^{\infty} a_n \phi_n.$$

$$L(u) = \sum a_n L(\phi_n) = -\sum a_n \lambda_n \sigma \phi_n.$$

$$\int_a^b u L(u)\,dx = -\int_a^b \sum a_n \phi_n \sum a_k \lambda_k \sigma \phi_k \,dx$$

$$= -\sum \lambda_n a_n^2 \int_a^b \phi_n^2 \sigma \,dx$$

$$R\varphi[u] = \frac{\sum_{n=1} \lambda_n a_n^2 \|\phi_n\|^2}{\sum a_n^2 \|\phi_n\|^2} = \geq \lambda_1$$

all $\lambda_n \geq \lambda_1$

$$\Rightarrow R\varphi[u] \geq \lambda_1$$

Choosing $\langle u, \phi_1 \rangle = 0$ and u satisfies the BCS.

we can prove that $\Rightarrow a_1 = 0$

$$R\psi[u] = \frac{\sum_{n=2} \lambda_n a_n^2 \| \|^2}{\sum_{n=2} a_n^2 \| \|^2} \geq \lambda_2 .$$

$$\lambda_2 \leq R\psi(u)$$

we can proceed similarly for higher eigenvalues.

5.8 Boundary conditions of the third Kind

Heat flow in a uniform rod satisfies

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Uniform vibration string solves

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

In either case we suppose that the left end is fixed, but the right end satisfies a homogeneous BC of the third kind

$$u(0,t) = 0$$
$$\frac{\partial u}{\partial x}(L,t) = -h\, u(L,t)$$

$h > 0$ physical condition (mathematically we have all possibilities).

wave string $\Rightarrow$ · If $h < 0$, the vibrating string has a destabilizing force at the right end,

heat rod $\Rightarrow$ - for the heat flow problem termal energy is being constantly put into the rod through the right end

After the separation of variables

~~heat flow~~ $u(x,t) = G(t)\ \phi(x)$

$\Rightarrow$ heat flow : $\dfrac{dG}{dt} = -\lambda k G \Rightarrow G = e^{-\lambda kt}$

vibrating string : $\dfrac{d^2G}{dt^2} = -\lambda c^2 G$

$$G = A\cos\sqrt{\lambda}ct + B\sin\sqrt{\lambda}ct$$

$$\lambda > 0$$

. both cases physically acceptable soln $\lambda > 0$

. Both cases the eigenvalue problem

PDE $\dfrac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad 0 < x < L$

BCS $\phi(0) = 0$

$\phi'(L) + h\,\phi(L) = 0$

where h is a given constant. The case

$h \geq 0$ physical

$h < 0$ unphysical

a) Positive eigenvalues $(\lambda > 0)$.

$$\phi(x) = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x$$

since $\phi(0) = 0 \Rightarrow C_1 = 0 \Rightarrow$

$$\phi(x) = C_2 \sin\sqrt{\lambda}\, x. \qquad 0 \leq x < L$$

Secon BC gives

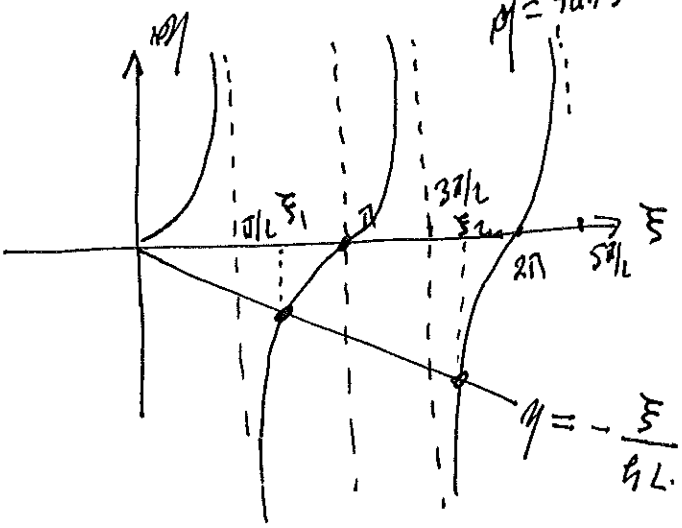
$$\sqrt{\lambda} \cos\sqrt{\lambda}\, L + h \sin\sqrt{\lambda}\, L = 0$$

or $$\tan\sqrt{\lambda}\, L = -\frac{\sqrt{\lambda}}{h}. \qquad (h \neq 0)$$

and $\cos(\sqrt{\lambda}\, L) \neq 0$. This is a transcendental equation ~~which~~ has infinitely many solutions for λ. depending on the sign of h.

$$\tan \xi = -\frac{\xi}{hL}, \qquad \xi = \sqrt{\lambda}\, L$$

$h > 0$



$\xi_1, \xi_2, \ldots$ are the solutions of $\tan\xi = -\frac{\xi}{hL}$

$$\frac{\pi}{2} < \xi_1 < \pi, \quad \frac{3\pi}{2} < \xi_2 < 2\pi, \quad \ldots, \quad \overset{n-1/2}{(\cancel{2n+\tfrac{1}{2}})}\pi < \xi_n < n\pi$$

$$\frac{\pi}{2} < \sqrt{\lambda_1}\,L < \pi, \quad \frac{3\pi}{2} < \sqrt{\lambda_2}\,L < \pi, \quad \cdots (n-1/2)\pi < \sqrt{\lambda_n}\,L < n\pi$$

$$\frac{(n-1/2)^2\pi^2}{L^2} < \lambda_n < \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, \ldots$$

as $n \to \infty$ $\quad \lambda_n \sim \dfrac{(n-\frac{1}{2})^2\pi^2}{L^2}$

$$\phi_n = \sin\sqrt{\lambda_n}\,x.$$

$$u(x,t) = \sum A_n e^{-\lambda_n k t} \sin\sqrt{\lambda_n}\,x$$ heat flow

$$= \sum{}' \sin\sqrt{\lambda_n}\,x\,(A_n \cos\sqrt{\lambda_n}\,ct + B_n \sin\sqrt{\lambda_n}\,ct)$$

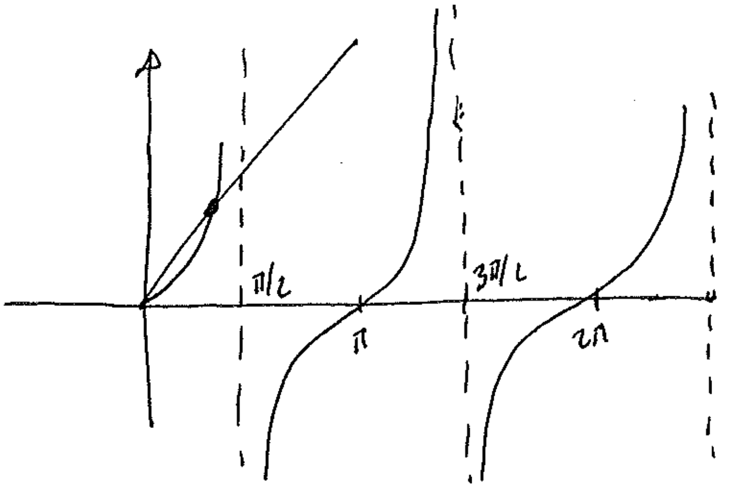
vibrating string.

$h = 0$ is the standard BC

$$\sqrt{\lambda_n}\,L = (n-\tfrac{1}{2})\pi; \quad n = 1, 2, \ldots$$

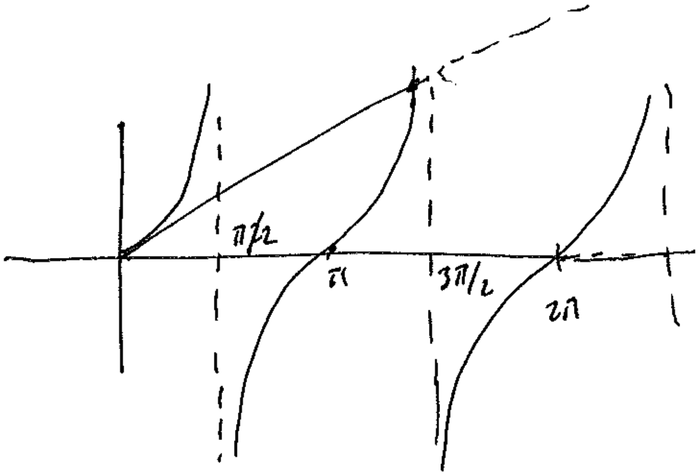
$$\lambda_n = \frac{(n-1/2)^2\pi^2}{L^2} \qquad n = 1, 2 \ldots$$

$h < 0$ (nonphysical ones): we have three cases i)



$\eta = \frac{\xi}{-hL}$, $0 > hL > -1$.

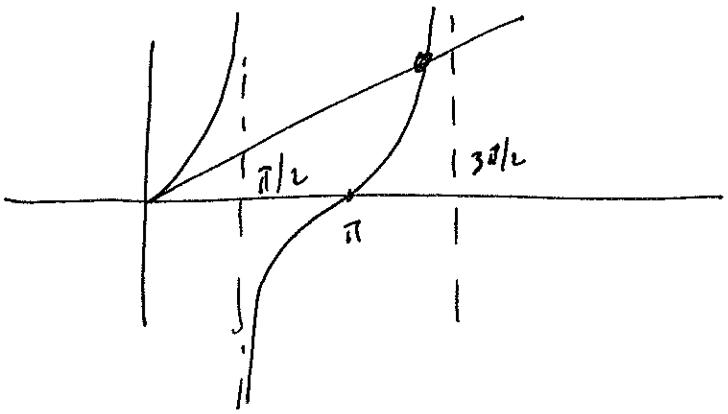
ii)



$\eta = \frac{\xi}{-hL}$, $hL = -1$.

- In all these cases there are infinitely many eigenvalues
- Large eigenvalues are closer to $(n-\frac{1}{2})\pi \sim \sqrt{\lambda_n}\,L$

iii)



$\eta = -\frac{\xi}{hL}$ $hL < -1$.

b) Negatif eigenvalues (unphysical)

$$s = -\lambda > 0$$

$$\phi'' = s\phi \Rightarrow \quad \phi(x) = c_1 \cosh\sqrt{s}x + c_2 \sinh\sqrt{s}x$$

$$\phi(0) = 0 \qquad c_1 = 0$$

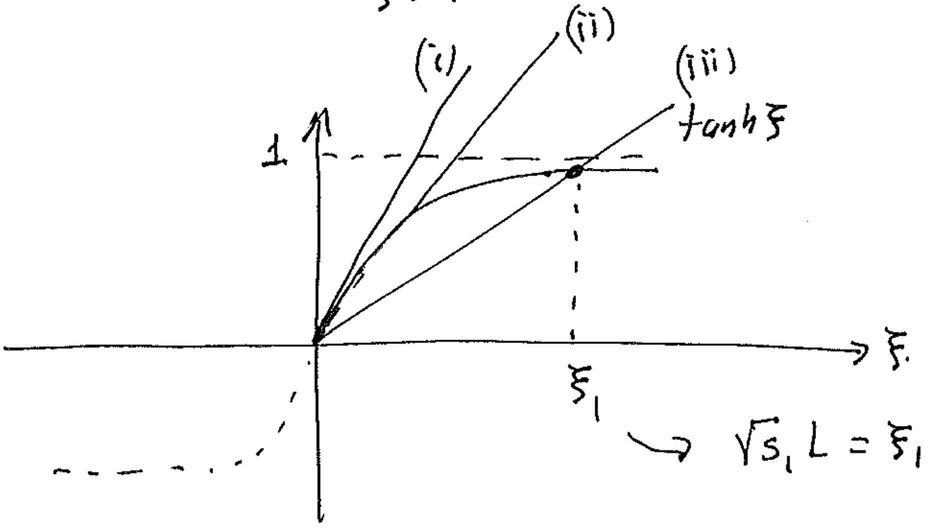
$$\phi(x) = c_2 \sinh\sqrt{s}x$$

2nd BC

$$\sqrt{s}\cosh\sqrt{s}L + h\sinh\sqrt{s}L = 0$$

$$\Rightarrow \quad \tanh\sqrt{s}L = -\frac{\sqrt{s}}{h}$$

$$\xi = \sqrt{s}L \qquad \tanh\xi = -\frac{\xi}{hL}$$



No root for $h > 0$

(i) $-1 < hL < 0$ (ii) $hL = -1$

(iii) $hL < -1$ one root

There is a negative eigenvalue when $hL < -1$ and infinitely many positive eigenvalues

c) zero eigenvalues:

$$\phi'' = 0 \Rightarrow \phi(x) = c_1 + c_2 x$$

$\phi(0) = 0 \qquad c_1 = 0$

$\phi(x) = c_2 x.$

$\phi'(L) + h\phi(L) = (1 + hL) c_2 = 0$

either $c_2 = 0$ trivial soln. or

$hL = -1.$

Hence for $hL = -1$,

$\lambda = 0$

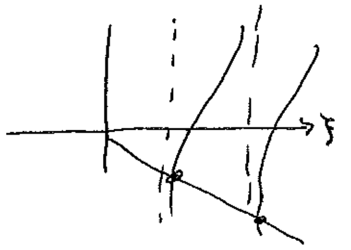
$\phi(x) = x.$

For other cases only trivial soln. possible.

$h > 0$

$\phi_n = \sin\sqrt{\lambda_n}x$

$\lambda > 0$ only



$\tan\xi = -\frac{\xi}{hL}$

$h = 0$

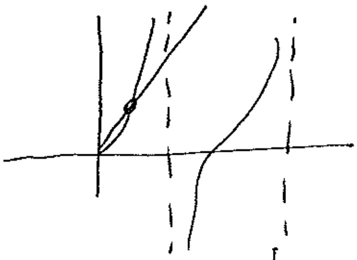
$\phi_n = \sin\sqrt{\lambda_n}x$

$$\lambda_n = \frac{(n-1/2)^2\pi^2}{L^2}$$

$-1 < hL < 0$

$\phi_n = \sin\sqrt{\lambda_n}x$

$\lambda > 0$ only



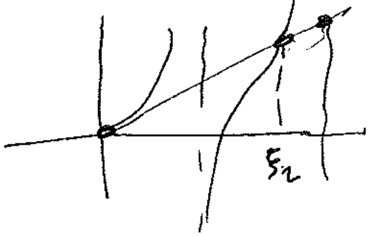
$hL = -1$

$\phi_n = \sin\sqrt{\lambda_n}x$

$\lambda > 0$

$n \geq 2$

$\phi_1 = x, \ \lambda = 0$

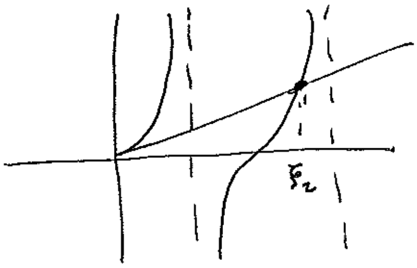


$hL < -1$

$\phi_n = \sin\sqrt{\lambda_n}x$

$\lambda > 0$

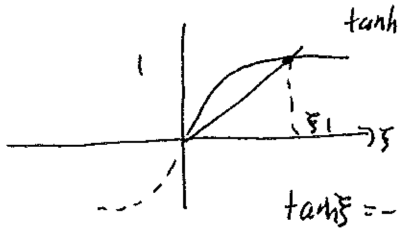
$n \geq 2$



$\phi_1 = \sinh\sqrt{s_1}x$

$\lambda_1 = -s_1 < 0$

$\lambda_1 = -\left(\frac{\xi_1}{L}\right)^2$



$\sqrt{s_1}L = \xi_1$

$\tanh\xi = -\frac{\xi}{hL}$

Table of eigenvalues and eigenfunctions

	$\lambda > 0$	$\lambda = 0$	$\lambda < 0$
$h > 0$	$\sin\sqrt{\lambda x}$		
$h = 0$	$\sin\sqrt{\lambda x}$		
$-1 < hL < 0$	$\sin\sqrt{\lambda}x$		
$hL = -1$	$\sin\sqrt{\lambda}\,x.$	x	
$hL < -1$	$\sin\sqrt{\lambda}x$		$\sinh\sqrt{s_1}\,x.$